Online Appendix for

‘Pyramidal Business Groups and Asymmetric Financial Frictions’

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Appendix

A Market Clearing Conditions

Capital market clears such that

$$\begin{aligned}
&\int \{a - c(z,a)\}\, dF(z,a) \\
&= \int_{o(z,a)=SA} \left\{k(z,a) + \mathbb{1}_{\sigma(z,a)>0} \cdot k^F\right\} \cdot \left\{1 - P^M(z,a)\right\}\, dF(z,a) \\
&+ \int_{o(z,a)=BG} \int_{\substack{o(z_2,a_2)=\{W,SA\} \\ w^M(z_2,a_2)\le \bar{w}^M(z,a|z_2)}} \left\{2k^F + w^M(z_2,a_2) + k_1^*(z,a|z_2,a_2) + k_2(z,a|z_2,a_2)\right\} \\
&\cdot P^{BG}(z_2(z,a),a_2)\, da_2 dF(z,a) \\
&+ \int_{o(z,a)=BG} \int_{\substack{o(z_2,a_2)=\{W,SA\} \\ w^M(z_2,a_2)> \bar{w}^M(z,a|z_2)}} \left\{k^F + k_1(z,a|k_2^C=0)\right\} \cdot P^{BG}(z_2(z,a),a_2)\, da_2 dF(z,a) \\
&+ \int_{o(z,a)=BG} \int_{o(z_2,a_2)=\{W,SA\}} \left\{k^F + k_1(z,a|k_2^C=0)\right\} \cdot \left\{1 - P^{BG}(z_2(z,a),a_2)\right\}\, da_2 dF(z,a);
\end{aligned} \tag{1}$$

and labor market clears such that

$$\begin{aligned}
&\int_{o(z,a)=W} \left\{1 - P^M(z,a)\right\} dF(z,a) \\
&= \int_{o(z,a)=SA} \int_{z'} \ell(z',k(z,a))\, dG(z'|z) \cdot \left\{1 - P^M(z,a)\right\}\, dF(z,a) \\
&+ \int_{o(z,a)=BG} \int_{\substack{o(z_2,a_2)=\{W,SA\} \\ w^M(z_2,a_2)\le \bar{w}^M(z,a|z_2)}} \left\{\int_{z'} \ell(z',k_1^*(z,a|z_2,a_2))\, dG(z'|z) + \int_{z_2'} \ell(z_2',k_2(z,a|z_2,a_2))\, dG(z_2'|z_2)\right\} \\
&\cdot P^{BG}(z_2(z,a),a_2)\, da_2 dF(z,a) \\
&+ \int_{o(z,a)=BG} \int_{\substack{o(z_2,a_2)=\{W,SA\} \\ w^M(z_2,a_2)> \bar{w}^M(z,a|z_2)}} \int_{z'} \ell(z',k_1(z,a|k_2^C=0))\, dG(z'|z) \cdot P^{BG}(z_2(z,a),a_2)\, da_2 dF(z,a) \\
&+ \int_{o(z,a)=BG} \int_{o(z_2,a_2)=\{W,SA\}} \int_{z'} \ell(z',k_1(z,a|k_2^C=0))\, dG(z'|z) \cdot \left\{1 - P^{BG}(z_2(z,a),a_2)\right\}\, da_2 dF(z,a)
\end{aligned} \tag{2}$$

where $G(z'|z)$ is a conditional cdf derived from the transition probability of managerial talents.

B Proof of Proposition 1

Let's define $\phi \in \left[\underline{\phi}, 1\right]$ and $\nu \le 1$ such that

$$k^C = \phi a, \quad k^D = \frac{1-\tau}{1+r}\nu \inf_{z',\delta'} \left[\pi(z',\delta'|z,k)\right]. \tag{3}$$

Then, a stand-alone entrepreneur running a publicly held corporation solves the following problem.

$$\mathcal{L}(z,a) = u\left((1-\phi)a - s\right) + \beta \mathbb{E}_{z',\delta'}\left[V(z',a')|z\right] + \lambda_s s + \lambda_\phi(\phi - \underline{\phi}) + \lambda_\nu(1-\nu) + \lambda_\sigma(\bar{\sigma} - \sigma) \tag{4}$$

where

$$
\begin{aligned}
a' &= (1+r)s + \tau\pi(z',\delta'|z,k) + (1-\sigma)(1-\tau)\left\{\pi(z',\delta'|z,k) - \nu \inf_{z',\delta'}\left[\pi(z',\delta'|z,k)\right]\right\} \\
k &= \phi a - k^F + \frac{1-\tau}{1+r}\left\{\sigma\mathbb{E}_{z',\delta'}\left[\pi(z',\delta'|z,k)\right] + (1-\sigma)\nu \inf_{z',\delta'}\left[\pi(z',\delta'|z,k)\right]\right\} \\
\underline{\phi} &= \frac{k^F}{a}.
\end{aligned}
\tag{5}
$$

To simplify notations, let's suppress arguments of functions and operators unless there is ambiguity. The corresponding Kuhn-Tucker conditions are as follows. For the optimal private saving, s,

$$
\begin{aligned}
\lambda_s s &= 0, \\
\lambda_s &= u'\left((1-\phi)a - s\right) - (1+r)\beta\mathbb{E}V_a \\
&\geq 0.
\end{aligned}
\tag{6}
$$

For the optimal private finance, $k^C = \phi a$,

$$
\begin{aligned}
\lambda_\phi(\phi - \underline{\phi}) &= 0, \\
\lambda_\phi &= a\underbrace{\left[u'\left((1-\phi)a - s\right) - (1+r)\beta\mathbb{E}V_a\right]}_{=\lambda_s} - a\beta\mathbb{E}\left[V_a \cdot \{-(1+r) + AB\}\right] \\
&\geq 0
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
A &\equiv \left[1 - \frac{1-\tau}{1+r}\left\{\sigma\mathbb{E}_{z',\delta'}\left[\frac{d}{dk}\pi(z',\delta'|z,k)\right] + (1-\sigma)\nu \inf_{z',\delta'}\left[\frac{d}{dk}\pi(z',\delta'|z,k)\right]\right\}\right]^{-1} \\
B &\equiv \tau\frac{d}{dk}\pi(z',\delta'|z,k) + (1-\sigma)(1-\tau)\left\{\frac{d}{dk}\pi(z',\delta'|z,k) - \nu \inf_{z',\delta'}\left[\frac{d}{dk}\pi(z',\delta'|z,k)\right]\right\}.
\end{aligned}
\tag{8}
$$

For the optimal external debt finance, $k^D = \frac{1-\tau}{1+r}\nu \inf \pi$,

$$
\begin{aligned}
\lambda_\nu(1-\nu) &= 0, \\
\lambda_\nu &= (1-\sigma)\frac{1-\tau}{1+r}\inf\pi\beta\mathbb{E}\left[V_a \cdot \{-(1+r) + AB\}\right] \\
&= \underbrace{(1-\sigma)\frac{1-\tau}{1+r}\inf\pi A}_{=\frac{dk}{d\nu}} \cdot \underbrace{\left\{\underbrace{\beta\mathbb{E}\left[V_a \cdot \{\mathbb{E}\pi' - (1+r)\}\right]}_{\text{Marginal Value of Expected Return}} - \underbrace{(1-\sigma+\sigma\tau)\beta\mathbb{E}\left[V_a \cdot \{\mathbb{E}\pi' - \pi'\}\right]}_{\text{Marginal Cost of Risk}}\right\}}_{\text{Marginal Expected Value of Investment}} \\
&\geq 0
\end{aligned}
\tag{9}
$$

where

$$
\pi' \equiv \frac{d}{dk}\pi(z',\delta'|z,k).
$$

Lastly, given $\sigma > 0$, the optimal external equity finance, $k^E = \frac{\sigma}{1+r}\left\{(1-\tau)\mathbb{E}\pi - (1+r)k^D\right\}$, satisfies the following conditions.

$$\lambda_\sigma(\bar{\sigma} - \sigma) = 0,$$

$$\lambda_\sigma = \underbrace{(1-\tau)\beta\mathbb{E}\left[V_a \cdot \{\mathbb{E}\pi - \pi\}\right]}_{\substack{=\beta\mathbb{E}\left[V_a \cdot \frac{da'}{d\sigma}\Big|_{dk(\nu,\sigma)=0}\right]>0 \\ \text{Marginal Value of Risk Sharing} \\ \text{Given the Fixed Amount of Capital } k}} + \underbrace{\underbrace{\frac{1-\tau}{1+r}\left(\mathbb{E}\pi - \nu \inf \pi\right) A}_{=\frac{dk}{d\sigma}} \cdot \beta\mathbb{E}\left[V_a \cdot \left\{\mathbb{E}\pi' - (1+r) - (1-\sigma+\sigma\tau)(\mathbb{E}\pi' - \pi')\right\}\right]}_{=|J|\cdot\lambda_\nu \text{ where } |J| = \left|\frac{d\nu}{d\sigma}\right|_{dk(\nu,\sigma)=0} = \frac{\mathbb{E}\pi - \nu \inf \pi}{(1-\sigma)\inf \pi}>0}$$

$$\geq 0 \tag{10}$$

Proof. From the Kuhn-Tucker condition for λ_ϕ,

$$\lambda_s = \frac{1}{a}\left\{\lambda_\phi + |J|\,\lambda_\nu\right\} \text{ where } |J| = \left|\frac{d\nu}{d\phi}\right|_{dk(\phi,\nu)=0} > 0.$$

Given the assumption that a firm is allowed to invest in a risk-free asset, the external debt finance k^D is only bounded above such that $\lambda_\nu \geq 0$. If $\lambda_\nu > 0$, $\lambda_s > 0$ and the optimal private saving is bounded below such that $s = 0$. If $\lambda_\nu = 0$, $\lambda_s = \frac{\lambda_\phi}{a}$ and the optimal private saving and the optimal private finance are undetermined because the marginal costs of them are aligned such that $\mathbb{1}_{\lambda_s>0} = \mathbb{1}_{\lambda_\phi>0}$. Thus, the zero private saving, $s = 0$, is weakly preferred and the optimization can be achieved by choosing $\{\phi, \nu, \sigma\}$ with $s = 0$.

Given Condition 1 and $\lambda_\nu \geq 0$, the marginal value of external equity finance is always greater than zero such that

$$\lambda_\sigma = (1-\tau)\beta\mathbb{E}\left[V_a \cdot \{\mathbb{E}\pi - \pi\}\right] + \left|\frac{d\nu}{d\sigma}\right|_{dk(\nu,\sigma)=0} \cdot \lambda_\nu$$

$$> 0.$$

Thus, given $\sigma > 0$, the optimal external equity finance is bounded above such that $\sigma = \bar{\sigma}$. □

Figure 1 shows that the optimal external equity finance is binding. Given the entrepreneur's managerial talent and wealth, (z, a), there is a downward sloping curve on which the marginal expected value of investment is zero such that $\lambda_\nu(\sigma, k(\sigma,\nu), a'(\sigma,\nu)|s,\phi) = 0$. From the Proposition 1, the marginal value of external equity finance is always positive on the curve such that $\lambda_\sigma|_{\lambda_\nu=0} = \beta\mathbb{E}\left[V_a \cdot \frac{da'}{d\sigma}\Big|_{dk=0}\right] = (1-\tau)\beta\mathbb{E}\left[V_a \cdot \{\mathbb{E}\pi - \pi\}\right] > 0$ because of the positive marginal benefit of risk sharing through external equity finance. The entrepreneur, thus, sells her firm's shares as many as possible until the constraint for the external equity finance is binding such that $\sigma = \bar{\sigma}_{SA}$.

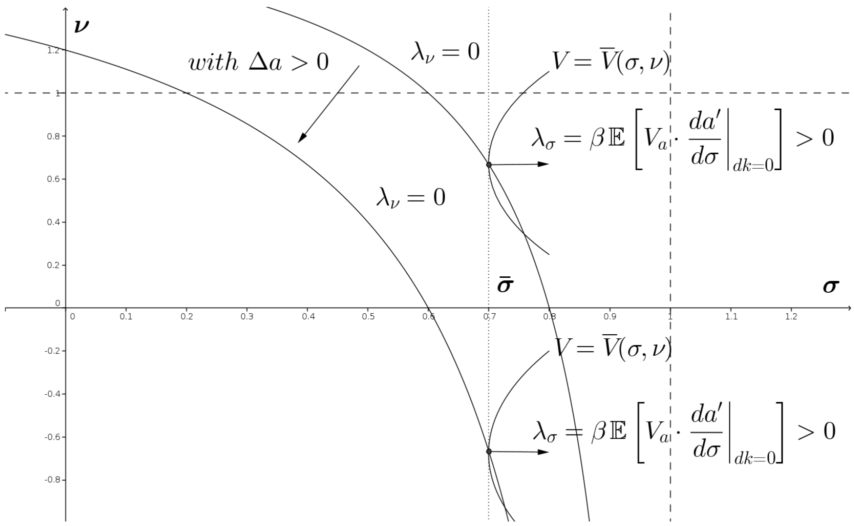

Figure 1: Risk Sharing and Binding External Equity Finance

Second, Figure 2 shows how the optimal private saving becomes zero. The risk-free investment opportunity keeps the marginal opportunity cost of private saving greater than or equal to that of private finance such that $a\lambda_s \geq \lambda_\phi \geq 0$. Given $a\lambda_s \geq \lambda_\phi$, the indifference curve $V = \overline{V}(\phi, s)$ cuts from below the line of constant marginal opportunity cost of private saving, $\lambda_s = \overline{\lambda}_s(c, a')$, which is achieved by $dc(s,\phi) = da'(s,\nu(s,\phi),k(\phi,\nu)) = dk(\phi,\nu(s,\phi)) = 0$. Thus, the indifference curve is pushed down until the borrowing constraint of an entrepreneur is binding such that $s = 0$.

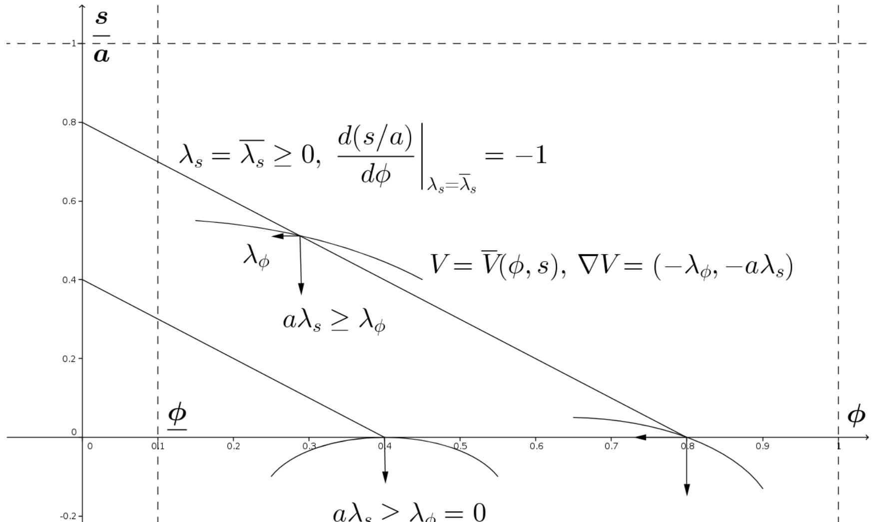

Figure 2: Non-Negative Marginal Expected Value of Investment and Binding Private Borrowing

C Proof of Proposition 2

Let's define $\phi \in \left[\underline{\phi}, 1\right]$, $\nu_1 \leq 1$, and $\nu_2 \leq 1$ such that

$$
\begin{aligned}
k_1^C &= \phi a, \\
k_1^D &= \frac{1-\tau}{1+r}\nu_1 \left[\inf_{z_1', \delta_1'} \left[\pi(z_1', \delta_1' | z_1, k_1^*)\right] + (1-\sigma_2)\left\{(1-\tau)\inf_{z_2', \delta_2'}\left[\pi(z_2', \delta_2'|z_2, k_2)\right] - (1+r)k_2^D\right\}\right], \\
k_2^D &= \frac{1-\tau}{1+r}\nu_2 \inf_{z_2', \delta_2'} \pi\left[(z_2', \delta_2'|z_2, k_2)\right].
\end{aligned}
\tag{11}
$$

Then, given (z_2, w^M), a business-group entrepreneur with (z_1, a) solves the following problem.

$$
\begin{aligned}
\mathcal{L}(z_1, a|z_2, w^M) = u\left((1-\phi)a - s\right) + \beta \mathbb{E}_{z_1', z_2', \delta_1', \delta_2'}\left[V(z_1', a')|z_1\right] + \lambda_s s + \lambda_\phi(\phi - \underline{\phi}) + \lambda_{k_2^C}\left(k_2^C - k^F - w_M\right) \\
+ \lambda_{\nu_1}(1-\nu_1) + \lambda_{\nu_2}(1-\nu_2) + \lambda_{\sigma_1}(\bar{\sigma} - \sigma_1) + \lambda_{\sigma_2}(\bar{\sigma} - \sigma_2)
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
a' &= (1+r)s + \tau\pi(z_1', \delta_1'|z_1, k_1^*) \\
&\quad + (1-\sigma_1)(1-\tau)\left[\pi(z_1', \delta_1'|z_1, k_1^*) - \nu_1\left\{\inf_{z_1', \delta_1'}\left[\pi(z_1', \delta_1'|z_1, k_1^*)\right] + (1-\sigma_2)(1-\nu_2)(1-\tau)\inf_{z_2', \delta_2'}\left[\pi(z_2', \delta_2'|z_2, k_2)\right]\right\}\right] \\
&\quad + \tau\pi(z_2', \delta_2'|z_2, k_2) + (1-\sigma_1+\sigma_1\tau)(1-\sigma_2)(1-\tau)\left\{\pi(z_2', \delta_2'|z_2, k_2) - \nu_2\inf_{z_2', \delta_2'}\left[\pi(z_2', \delta_2'|z_2, k_2)\right]\right\} \\
k_1^* &= \phi a - k^F - k_2^C + \frac{1-\tau}{1+r}\left\{\sigma_1 \mathbb{E}_{z_1', \delta_1'}\left[\pi(z_1', \delta_1'|z_1, k_1^*)\right] + (1-\sigma_1)\nu_1\inf_{z_1', \delta_1'}\left[\pi(z_1', \delta_1'|z_1, k_1^*)\right]\right\} \\
&\quad + \frac{(1-\tau)^2(1-\sigma_2)}{1+r}\left[\sigma_1\left\{\mathbb{E}_{z_2', \delta_2'}\left[\pi(z_2', \delta_2'|z_2, k_2)\right] - \nu_2\inf_{z_2', \delta_2'}\left[\pi(z_2', \delta_2'|z_2, k_2)\right]\right\} + (1-\sigma_1)\nu_1(1-\nu_2)\inf_{z_2', \delta_2'}\left[\pi(z_2', \delta_2'|z_2, k_2)\right]\right] \\
k_2 &= k_2^C - k^F - w^M + \frac{1-\tau}{1+r}\left\{\sigma_2 \mathbb{E}_{z_2', \delta_2'}\left[\pi(z_2', \delta_2'|z_2, k_2)\right] + (1-\sigma_2)\nu_2\inf_{z_2', \delta_2'}\left[\pi(z_2', \delta_2'|z_2, k_2)\right]\right\}.
\end{aligned}
\tag{13}
$$

The corresponding Kuhn-Tucker conditions are as follows. Let's suppress arguments of functions and operators for simplicity unless there is ambiguity. For the optimal private saving, s,

$$
\begin{aligned}
\lambda_s s &= 0 \\
\lambda_s &= u'\left((1-\phi)a - s\right) - (1+r)\beta\mathbb{E}V_a \\
&\geq 0.
\end{aligned}
\tag{14}
$$

For the optimal private finance of Firm 1, $k_1^C = \phi a$,

$$
\begin{aligned}
\lambda_\phi(\phi - \underline{\phi}) &= 0 \\
\lambda_\phi &= a\underbrace{\left[u'\left((1-\phi)a - s\right) - (1+r)\beta\mathbb{E}V_a\right]}_{=\lambda_s} - a\beta\mathbb{E}\left[V_a \cdot \{-(1+r) + A_1 B_1\}\right] \\
&\geq 0
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
A_1 &\equiv \left[1 - \frac{1-\tau}{1+r}\left\{\sigma_1 \mathbb{E}_{z_1',\delta_1'}\pi_1' + (1-\sigma_1)\nu_1 \inf_{z_1',\delta_1'} \pi_1'\right\}\right]^{-1} \\
B_1 &\equiv \tau\pi_1' + (1-\sigma_1)(1-\tau)(\pi_1' - \nu_1 \inf \pi_1') \\
\pi_1' &\equiv \frac{d}{dk_1^*}\pi_1(z_1', \delta_1'|z_1, k_1^*).
\end{aligned} \tag{16}
$$

For the optimal external debt finance of Firm 1, $k_1^D = \frac{1-\tau}{1+r}\nu_1\left[\inf \pi_1 + (1-\sigma_2)(1-\nu_2)(1-\tau)\inf \pi_2\right]$,

$$
\begin{aligned}
&\lambda_{\nu_1}(1-\nu_1) = 0 \\
&\quad \lambda_{\nu_1} = (1-\sigma_1)\frac{1-\tau}{1+r}\left\{\inf \pi_1 + (1-\sigma_2)(1-\tau)(1-\nu_2)\inf \pi_2\right\} \cdot \beta\mathbb{E}\left[V_a \cdot \{-(1+r) + A_1 B_1\}\right] \\
&\quad = \underbrace{(1-\sigma_1)\frac{1-\tau}{1+r}\left\{\inf \pi_1 + (1-\sigma_2)(1-\tau)(1-\nu_2)\inf \pi_2\right\} A_1}_{=\frac{dk_1^*}{d\nu_1}} \\
&\quad \cdot \underbrace{\left\{\underbrace{\beta\mathbb{E}\left[V_a \cdot \{\mathbb{E}\pi_1' - (1+r)\}\right]}_{\text{Marginal Value of Expected Return}} - \underbrace{(1-\sigma_1+\sigma_1\tau)\beta\mathbb{E}\left[V_a \cdot \{\mathbb{E}\pi_1' - \pi_1'\}\right]}_{\text{Marginal Cost of Risk}}\right\}}_{\text{Marginal Expected Value of Investment through Firm 1}} \\
&\quad \geq 0.
\end{aligned} \tag{17}
$$

For the optimal external equity finance of Firm 1,

$$
k_1^E = \frac{\sigma_1}{1+r}\left[(1-\tau)\mathbb{E}\pi_1 + (1-\tau)(1-\sigma_2)\left\{(1-\tau)\mathbb{E}\pi_2 - (1+r)k_2^D\right\} - (1+r)k_1^D\right],
$$

$$
\begin{aligned}
&\lambda_{\sigma_1}(\bar{\sigma} - \sigma_1) = 0 \\
&\quad \lambda_{\sigma_1} = \underbrace{(1-\tau)\beta\mathbb{E}\left[V_a \cdot \{\mathbb{E}\pi_1 - \pi_1\}\right] + (1-\tau)^2(1-\sigma_2)\beta\mathbb{E}\left[V_a \cdot \{\mathbb{E}\pi_2 - \pi_2\}\right]}_{\substack{=\beta\mathbb{E}\left[V_a \cdot \frac{da'}{d\sigma_1}\Big|_{dk_1^*(\sigma_1,\nu_1)=0}\right]>0 \\ \text{Marginal Value of Risk Sharing Through Firm 1 Given Capital } (k_1^*, k_2)}} \\
&\quad + \underbrace{\underbrace{\left[\frac{1-\tau}{1+r}\left\{\mathbb{E}\pi_1 - \nu_1\left(\inf \pi_1 + (1-\sigma_2)(1-\tau)(1-\nu_2)\inf \pi_2\right)\right\} + \frac{(1-\tau)^2}{1+r}(1-\sigma_2)\left\{\mathbb{E}\pi_2 - \nu_2 \inf \pi_2\right\}\right] A_1}_{=\frac{dk_1^*}{d\sigma_1}} \cdot \left\{\beta\mathbb{E}\left[V_a \cdot \left\{\mathbb{E}\pi_1' - (1+r)\right\}\right] - (1-\sigma_1+\sigma_1\tau)\beta\mathbb{E}\left[V_a \cdot \left\{\mathbb{E}\pi_1' - \pi_1'\right\}\right]\right\}}_{=|J|\cdot\lambda_{\nu_1} \text{ where } |J| = \left|\frac{d\nu_1}{d\sigma_1}\right|_{dk_1^*(\sigma_1,\nu_1)=0}} \\
&\quad \geq 0.
\end{aligned} \tag{18}
$$

For the optimal internal equity finance from Firm 1 to Firm 2, k_2^C,

$$
\begin{aligned}
&\lambda_{k_2^C}(k_2^C - k^F - w^M) = 0 \\
&\qquad \lambda_{k_2^C} = \beta\mathbb{E}\left[V_a \cdot \{A_1 B_1 - A_{12}A_2A_1B_1 - A_2B_2\}\right] \\
&\qquad \geq 0
\end{aligned} \tag{19}
$$

where

$$
\begin{aligned}
A_{12} &\equiv \frac{(1-\tau)^2(1-\sigma_2)}{1+r}\left\{\sigma_1\left(\mathbb{E}_{z_2',\delta_2'}\pi_2' - \nu_2 \inf_{z_2',\delta_2'} \pi_2'\right) + (1-\sigma_1)\nu_1(1-\nu_2) \inf_{z_2',\delta_2'} \pi_2'\right\} \\
A_2 &\equiv \left[1 - \frac{1-\tau}{1+r}\left\{\sigma_2 \mathbb{E}_{z_2',\delta_2'}\pi_2' + (1-\sigma_2)\nu_2 \inf_{z_2',\delta_2'} \pi_2'\right\}\right]^{-1} \\
B_2 &\equiv \tau\pi_2' + (1-\sigma_1+\sigma_1\tau)(1-\sigma_2)(1-\tau)(\pi_2' - \nu_2 \inf_{z_2',\delta_2'} \pi_2') - (1-\tau)^2(1-\sigma_1)(1-\sigma_2)\nu_1(1-\nu_2) \inf_{z_2',\delta_2'} \pi_2'.
\end{aligned} \tag{20}
$$

For the optimal external debt finance of Firm 2, $k_2^D = \frac{1-\tau}{1+r}\nu_2 \inf \pi_2$,

$$
\begin{aligned}
&\lambda_{\nu_2}(1-\nu_2) = 0 \\
&\qquad \lambda_{\nu_2} = \underbrace{\frac{1-\tau}{1+r}(1-\sigma_2)\inf\pi_2 \left\{\tau + (1-\tau)(1-\sigma_1)(1-\nu_1)\right\} A_1}_{=\left.\frac{dk_1^*}{d\nu_2}\right|_{dk_2(k_2^C,\nu_2)=0} \quad \to 0 \text{ as } \tau\to 0 \text{ with } \nu_1=1} \cdot \underbrace{\beta\mathbb{E}\left[V_a \cdot \left\{(\mathbb{E}\pi_1' - (1+r)) - (1-\sigma_1+\sigma_1\tau)(\mathbb{E}\pi_1' - \pi_1')\right\}\right]}_{\text{Marginal Expected Value of Investment through Firm 1}} \\
&\qquad\qquad \underbrace{-\frac{1-\tau}{1+r}(1-\sigma_2)\inf\pi_2}_{=\left.\frac{dk_2^C}{d\nu_2}\right|_{dk_2=0}} \cdot \underbrace{\beta\mathbb{E}\left[V_a \cdot \{A_1B_1 - A_{12}A_2A_1B_1 - A_2B_2)\right]}_{=\lambda_{k_2^C}} \\
&\qquad\qquad \geq 0.
\end{aligned} \tag{21}
$$

For the optimal external equity finance of Firm 2, $k_2^E = \frac{\sigma_2}{1+r}\left\{(1-\tau)\mathbb{E}\pi_2 - (1+r)k_2^D\right\}$,

$$
\begin{aligned}
&\lambda_{\sigma_2}(\bar{\sigma} - \sigma_2) = 0 \\
&\lambda_{\sigma_2} = \underbrace{\beta\mathbb{E}\left[V_a \cdot \frac{1-\tau}{1+r}\left(\mathbb{E}\pi_2 - \nu_2\inf\pi_2\right)\left\{\left(-1 + A_{12}A_2\right)A_1B_1 + A_2B_2 + \{\tau + (1-\tau)(1-\sigma_1)(1-\nu_1)\}\{-(1+r) + A_1B_1\}\right\}\right]}_{=|J|\cdot\lambda_{\nu_2} \text{ where } |J| = \left|\frac{d\nu_2}{d\sigma_2}\right|_{dk_2(\nu_2,\sigma_2)=0} = \frac{\mathbb{E}\pi_2 - \nu_2\inf\pi_2}{(1-\sigma_2)\inf\pi_2}} \\
&\quad + \underbrace{\frac{(1-\tau)^2}{1+r}(1-\sigma_1)\nu_1(\mathbb{E}\pi_2 - \inf\pi_2)A_1}_{=\left.\frac{dk_1^*}{d\sigma_2}\right|_{dk_2(\nu_2,\sigma_2)=0}} \cdot \underbrace{\beta\mathbb{E}\left[V_a \cdot \left\{(\mathbb{E}\pi_1' - (1+r)) - (1-\sigma_1+\sigma_1\tau)(\mathbb{E}\pi_1' - \pi_1')\right\}\right]}_{\text{Marginal Expected Value of Investment through Firm 1}} \\
&\quad + \qquad \underbrace{\beta\mathbb{E}\left[V_a \cdot (1-\tau)(1-\sigma_1+\sigma_1\tau)\left\{\mathbb{E}\pi_2 - \pi_2\right\}\right]}_{\substack{=\beta\mathbb{E}\left[V_a \cdot \left.\frac{da'}{d\sigma_2}\right|_{dk_1^*=dk_2=0}\right] \\ \text{Marginal Value of Risk Sharing Through Firm 2 Given Capital } (k_1^*, k_2)}} \\
&\geq 0.
\end{aligned} \tag{22}
$$

Proof. From the Kuhn-Tucker condition for λ_ϕ,

$$
\lambda_s = \frac{1}{a}\left\{\lambda_\phi + |J|\,\lambda_{\nu_1}\right\} \text{ where } |J| = \left|\frac{d\nu_1}{d\phi}\right|_{dk_1^*=0} > 0.
$$

Given the assumption that firms are allowed to invest in a risk-free asset, the external debt finance of Firm 1 is only bounded above such that $\lambda_{\nu_1} \geq 0$. If $\lambda_{\nu_1} > 0$, $\lambda_s > 0$ and the optimal private saving is bounded below such that $s = 0$. If $\lambda_{\nu_1} = 0$, $\mathbb{1}_{\lambda_s} = \mathbb{1}_{\lambda_\phi}$ and the optimal private saving and the optimal private finance are undetermined unless they are binding together. Thus, the zero private saving is weakly preferred and the optimization can be achieved with $s = 0$.

From the Kuhn-Tucker condition for λ_{ν_2},

$$\lambda_{\nu_1} = C \cdot \lambda_{k_2^C} + D \cdot \lambda_{\nu_2}, \quad C, D > 0 \text{ given } \tau > 0.$$

Since firms are allowed to invest in a risk-free asset, the external debt finance of Firm 2 is only bounded above such that $\lambda_{\nu_2} \geq 0$. If $\lambda_{\nu_2} > 0$, $\lambda_{\nu_1} > 0$ and the optimal external debt finance of Firm 1 is bounded above such that $\nu_1 = 1$. If $\lambda_{\nu_2} = 0$, $\mathbb{1}_{\lambda_{\nu_1}} = \mathbb{1}_{\lambda_{k_2^C}}$ and the optimal external debt finance of Firm 1 and the optimal internal equity finance are undetermined unless they are binding together. Thus, the full external debt finance of Firm 1 is weakly preferred and the optimization can be achieved with $\nu_1 = 1$.

Given Condition 2 and $\lambda_{\nu_1}, \lambda_{\nu_2} \geq 0$, the marginal values of external equity finance of Firm 1 and Firm 2 are always greater than zero such that,

$$\lambda_{\sigma_1} \geq \beta \mathbb{E}\left[V_a \cdot \left.\frac{da'}{d\sigma_1}\right|_{dk_1^* = 0} \right] > 0,$$

$$\lambda_{\sigma_2} \geq \beta \mathbb{E}\left[V_a \cdot \left.\frac{da'}{d\sigma_2}\right|_{dk_1^* = dk_2 = 0} \right] > 0.$$

Thus, the optimal external equity finance is binding such that $(\sigma_1, \sigma_2) = (\bar{\sigma}, \bar{\sigma})$. □

The intuition of Proposition 2 is similar to that of Proposition 1. Given the non-negative value of investment, the risk sharing motive makes an entrepreneur to sell both her shares of Firm 1 and Firm 1's shares of Firm 2 as many as possible. Thus, the constraints for the external equity finance of Firm 1 and Firm 2 are binding.

Moreover, the risk-free investment opportunity of firms makes an entrepreneur to take advantage of external debt finance of Firm 1 and carry it over into Firm 2. It is entrepreneur's relegated saving in the sense that the risk-free cash flow of Firm 2 is diverted out to the entrepreneur due to financial frictions. Note that financial frictions are required to link λ_{ν_1} and λ_{ν_2}. If $\tau = 0$, the Kuhn-Tucker conditions are collapsed into $\lambda_{\nu_2} = \lambda_{k_2^C} = 0$ regardless of λ_{ν_1} and the full external debt finance of Firm 1 is not guaranteed anymore.

The following Figure 3 shows that the borrowing constraint for Firm 1 is binding. The risk-free investment opportunity of Firm 2 keeps the marginal value of external debt finance of Firm 1 is greater than or equal to the marginal opportunity cost of internal equity finance such that $\lambda_{\nu_1} \geq C\lambda_{k_2^C} \geq 0$. Given $\lambda_{\nu_1} \geq C\lambda_{k_2^C}$, the indifference curve $V = \overline{V}(\nu_1, k_2^C)$ cuts from above the curve of constant marginal value of external debt finance of Firm 1, $\lambda_{\nu_1} = \overline{\lambda}_{\nu_1}(k_1^*, a')$, which is achieved by $dk_1^*(\nu_1, k_2^C, \nu_2(\nu_1, k_2^C), k_2(k_2^C, \nu_2)) = dk_2(k_2^C, \nu_2(\nu_1, k_2^C)) = da'(\nu_1, \nu_2(\nu_1, k_2^C), k_1^*, k_2) = 0$. Thus, the indifference curve is pushed up until the borrowing constraint of Firm 1 is binding such that $\nu_1 = 1$.

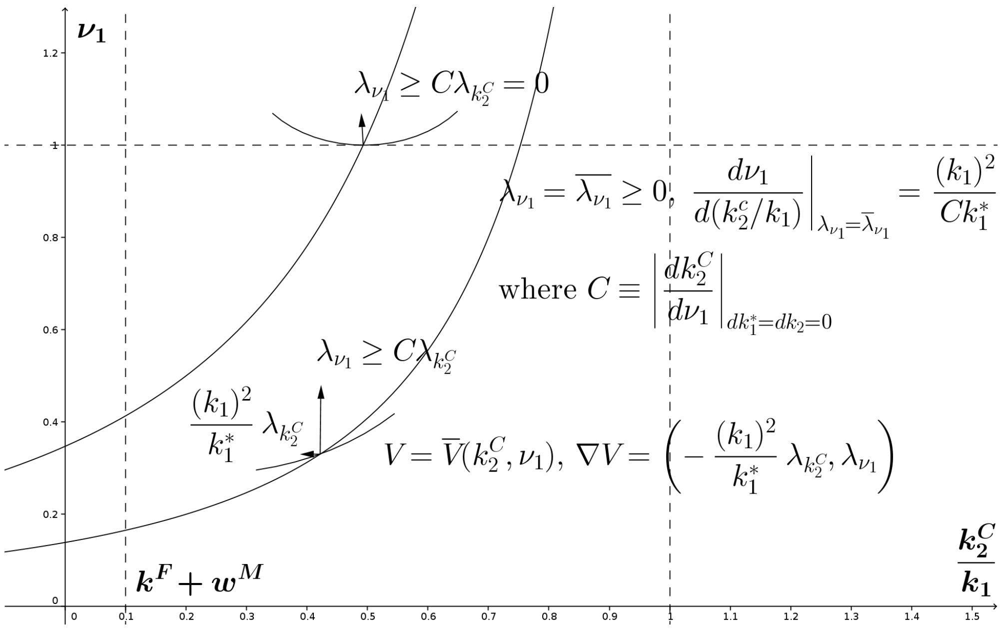

Figure 3: Non-negative Marginal Expected Value of Investment and Binding External Debt Finance of Firm 1

D Some Algebra

The following algebra is omitted in the above entrepreneur's problem for brevity.

A Stand-Alone Entrepreneur's Problem

From $\lambda_\nu \geq 0$,

$$\begin{aligned}
-(1+r) + AB &= A\left[-(1+r)A^{-1} + B\right] \\
&= A\left[-(1+r) + (1-\tau)\left\{\sigma\mathbb{E}\pi' + (1-\sigma)\nu\inf\pi'\right\} + (1-\sigma+\sigma\tau)\pi' - (1-\sigma)(1-\tau)\nu\inf\pi'\right] \\
&= A\left[\mathbb{E}\pi' - (1+r) - (1-\sigma+\sigma\tau)(\mathbb{E}\pi' - \pi')\right]
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
A &\equiv \left[1 - \frac{1-\tau}{1+r}\left\{\sigma\mathbb{E}\pi' + (1-\sigma)\nu\inf\pi'\right\}\right]^{-1} \\
B &\equiv \tau\pi' + (1-\sigma)(1-\tau)\left\{\pi' - \nu\inf\pi'\right\}.
\end{aligned} \tag{24}$$

From $\lambda_\sigma \geq 0$,

$$\begin{aligned}
A^{-1}dk(\nu,\sigma) &= \frac{1-\tau}{1+r}(1-\sigma)\inf\pi d\nu + \frac{1-\tau}{1+r}\left(\mathbb{E}\pi - \nu\inf\pi\right)d\sigma \\
\frac{d\nu}{d\sigma}\Big|_{dk(\nu,\sigma)=0} &= -\frac{\mathbb{E}\pi - \nu\inf\pi}{(1-\sigma)\inf\pi}
\end{aligned} \tag{25}$$

and

$$\begin{aligned} da'(\nu,\sigma)\big|_{dk=0} &= -(1-\sigma)(1-\tau)\inf \pi d\nu - (1-\tau)(\pi - \nu \inf \pi) d\sigma \\ \left.\frac{da'(\nu,\sigma)}{d\sigma}\right|_{dk=0} &= -(1-\sigma)(1-\tau)\inf \pi \cdot \left.\frac{d\nu}{d\sigma}\right|_{dk=0} - (1-\tau)\left(\pi - \nu \inf \pi\right) \\ &= (1-\tau)(\mathbb{E}\pi - \nu \inf \pi) - (1-\tau)(\pi - \nu \inf \pi) \\ &= (1-\tau)(\mathbb{E}\pi - \pi). \end{aligned} \tag{26}$$

In the proof of Proposition 1,

$$\begin{aligned} A^{-1} dk(\phi,\nu) &= a d\phi + \frac{1-\tau}{1+r}(1-\sigma)\inf \pi d\nu \\ |J| &= \left|\frac{d\nu}{d\phi}\right|_{dk(\phi,\nu)=0} \\ &= \frac{a}{\frac{1-\tau}{1+r}(1-\sigma)\inf \pi}. \end{aligned} \tag{27}$$

The line of constant marginal opportunity cost of private saving, $\lambda_s = \lambda_s(c(s,\phi), a'(s,\nu(s,\phi), k(\phi,\nu))|\sigma)$, is derived by solving for the following system of equations

$$\begin{aligned} dc(s,\phi) &= -ds - a d\phi = 0 \\ A^{-1} dk(\phi,\nu) &= a d\phi + \frac{1-\tau}{1+r}(1-\sigma)\inf \pi d\nu = 0 \\ da'(s,\nu)\big|_{dk=0} &= (1+r)ds - (1-\sigma)(1-\tau)\inf \pi d\nu = 0 \end{aligned} \tag{28}$$

such that

$$a d\phi = -ds = -\frac{1-\tau}{1+r}(1-\sigma)\inf \pi d\nu. \tag{29}$$

Note that $da' = 0$ is redundant with $dc = dk = 0$.

A Business-Group Entrepreneur's Problem

From $\lambda_{\sigma_1} \geq 0$,

$$\begin{aligned} A_1^{-1} dk_1^*(\nu_1,\sigma_1)\big|_{dk_2=0} &= \left\{\frac{1-\tau}{1+r}(1-\sigma_1)\inf \pi_1 + \frac{(1-\tau)^2(1-\sigma_2)}{1+r}(1-\sigma_1)(1-\nu_2)\inf \pi_2\right\} d\nu_1 \\ &+ \left\{\frac{1-\tau}{1+r}(\mathbb{E}\pi_1 - \nu_1 \inf \pi_1) + \frac{(1-\tau)^2(1-\sigma_2)}{1+r}(\mathbb{E}\pi_2 - \nu_2 \inf \pi_2) - \frac{(1-\tau)^2(1-\sigma_2)}{1+r}\nu_1(1-\nu_2)\inf \pi_2\right\} d\sigma_1 \\ da'(\nu_1,\sigma_1)\big|_{dk_1^*=dk_2=0} &= \left\{-(1-\sigma_1)(1-\tau)\inf \pi_1 - (1-\tau)^2(1-\sigma_1)(1-\sigma_2)(1-\nu_2)\inf \pi_2\right\} d\nu_1 \\ &+ \left\{-(1-\tau)(\pi_1 - \nu_1 \inf \pi_1) + (1-\tau)^2(1-\sigma_2)\nu_1(1-\nu_2)\inf \pi_2 - (1-\tau)^2(1-\sigma_2)(\pi_2 - \nu_2 \inf \pi_2)\right\} d\sigma_1. \end{aligned} \tag{30}$$

Adding to the bottom equation the upper equation multiplied by $(1+r)$ with taking $dk_1^* = 0$,

$$\begin{aligned} da'\big|_{dk_1^*=dk_2=0} &= \left\{(1-\tau)(\mathbb{E}\pi_1 - \pi_1) + (1-\tau)^2(1-\sigma_2)(\mathbb{E}\pi_2 - \pi_2)\right\} d\sigma_1 \\ \left.\frac{da'}{d\sigma_1}\right|_{dk_1^*=dk_2=0} &= \left\{(1-\tau)(\mathbb{E}\pi_1 - \pi_1) + (1-\tau)^2(1-\sigma_2)(\mathbb{E}\pi_2 - \pi_2)\right\}. \end{aligned} \tag{31}$$

From λ_{ν_2},

$$\begin{aligned} A_2^{-1} dk_2(k_2^C, \nu_2) &= dk_2^C + \frac{1-\tau}{1+r}(1-\sigma_2)\inf \pi_2 d\nu_2 \\ \left.\frac{dk_2^C}{d\nu_2}\right|_{dk_2(k_2^C,\nu_2)=0} &= -\frac{1-\tau}{1+r}(1-\sigma_2)\inf \pi_2 \end{aligned} \tag{32}$$

and

$$\begin{aligned} A_1^{-1} dk_1^*(k_2^C, \nu_2)\big|_{dk_2=0} &= -dk_2^C + \frac{(1-\tau)^2(1-\sigma_2)}{1+r}\inf \pi_2 \left\{-\sigma_1 - (1-\sigma_1)\nu_1\right\} d\nu_2 \\ \left.\frac{dk_1^*(k_2^C,\nu_2)}{d\nu_2}\right|_{dk_2=0} &= -A_1 \left.\frac{dk_2^C}{d\nu_2}\right|_{dk_2=0} - \frac{(1-\tau)^2(1-\sigma_2)}{1+r}\inf \pi_2 \left\{\sigma_1 + (1-\sigma_1)\nu_1\right\} A_1 \\ &= \frac{1-\tau}{1+r}(1-\sigma_2)\inf \pi_2 \left\{\tau + (1-\tau)(1-\sigma_1)(1-\nu_1)\right\} A_1. \end{aligned} \tag{33}$$

From λ_{σ_2},

$$\begin{aligned} A_1^{-1} dk_1^*(\nu_2, \sigma_2)\big|_{dk_2=0} &= -\frac{(1-\tau)^2(1-\sigma_2)}{1+r}\left\{\sigma_1 + (1-\sigma_1)\nu_1\right\}\inf \pi_2 d\nu_2 \\ &\quad - \frac{(1-\tau)^2}{1+r}\left\{\sigma_1(\mathbb{E}\pi_2 - \nu_2 \inf \pi_2) + (1-\sigma_1)\nu_1(1-\nu_2)\inf \pi_2\right\} d\sigma_2 \\ A_2^{-1} dk_2(\nu_2, \sigma_2) &= \frac{1-\tau}{1+r}(1-\sigma_2)\inf \pi_2 d\nu_2 + \frac{1-\tau}{1+r}(\mathbb{E}\pi_2 - \nu_2 \inf \pi_2) d\sigma_2. \end{aligned} \tag{34}$$

Adding to the upper equation the bottom equation multiplied by $(1-\tau)\left\{\sigma_1 + (1-\sigma_1)\nu_1\right\}$ with taking $dk_2 = 0$,

$$\begin{aligned} A_1^{-1} dk_1^*(\nu_2, \sigma_2)\big|_{dk_2=0} &= \frac{(1-\tau)^2}{1+r}\left[(\mathbb{E}\pi_2 - \nu_2 \inf \pi_2)\left\{-\sigma_1 + \sigma_1 + (1-\sigma_1)\nu_1\right\} - (1-\sigma_1)\nu_1(1-\nu_2)\inf \pi_2\right] d\sigma_2 \\ &= \frac{(1-\tau)^2(1-\sigma_1)\nu_1}{1+r}(\mathbb{E}\pi_2 - \inf \pi_2) d\sigma_2 \\ \left.\frac{dk_1^*(\nu_2, \sigma_2)}{d\sigma_2}\right|_{dk_2=0} &= \frac{(1-\tau)^2(1-\sigma_1)\nu_1}{1+r}(\mathbb{E}\pi_2 - \inf \pi_2) A_1. \end{aligned} \tag{35}$$

By adding up the following two equations with taking $dk_1^* = dk_2 = 0$,

$$\begin{aligned} dk_1^*(k_2^C, \nu_2, \sigma_2) &= -dk_2^C - \frac{(1-\tau)^2}{1+r}\left\{\sigma_1(\mathbb{E}\pi_2 - \nu_2 \inf \pi_2) + (1-\sigma_1)\nu_1(1-\nu_2)\inf \pi_2\right\} d\sigma_2 \\ &\quad + \frac{(1-\tau)^2(1-\sigma_2)}{1+r}\left\{-\sigma_1 \inf \pi_2 - (1-\sigma_1)\nu_1 \inf \pi_2\right\} d\nu_2 \\ dk_2(k_2^C, \nu_2, \sigma_2) &= dk_2^C + \frac{1-\tau}{1+r}(\mathbb{E}\pi_2 - \nu_2 \inf \pi_2) d\sigma_2 + \frac{1-\tau}{1+r}(1-\sigma_2)\inf \pi_2 d\nu_2, \end{aligned} \tag{36}$$

we can derive

$$\left.\frac{d\nu_2}{d\sigma_2}\right|_{dk_1^*=dk_2=0} = \frac{-(1-\sigma_1+\sigma_1\tau)(\mathbb{E}\pi_2 - \nu_2 \inf \pi_2) + (1-\tau)(1-\sigma_1)\nu_1(1-\nu_2)\inf \pi_2}{(1-\sigma_2)\inf \pi_2 \left\{\tau + (1-\tau)(1-\sigma_1)(1-\nu_1)\right\}}. \tag{37}$$

Then, by substituting for $\left.\frac{d\nu_2}{d\sigma_2}\right|_{dk_1^*=dk_2=0}$,

$$
\begin{aligned}
\left.\frac{da'(\nu_2,\sigma_2)}{d\sigma_2}\right|_{dk_1^*=dk_2=0} &= \left\{(1-\tau)^2(1-\sigma_1)\nu_1 - (1-\sigma_1+\sigma_1\tau)(1-\tau)\right\}(1-\sigma_2)\inf\pi_2 \cdot \left.\frac{d\nu_2}{d\sigma_2}\right|_{dk_1^*=dk_2=0} \\
&\quad + (1-\tau)^2(1-\sigma_1)\nu_1(1-\nu_2)\inf\pi_2 - (1-\sigma_1+\sigma_1\tau)(1-\tau)(\pi_2-\nu_2\inf\pi_2) \\
&= -(1-\tau)^2(1-\sigma_1)\nu_1(1-\nu_2)\inf\pi_2 + (1-\sigma_1+\sigma_1\tau)(1-\tau)(\mathbb{E}\pi_2-\nu_2\inf\pi_2) \\
&\quad + (1-\tau)^2(1-\sigma_1)\nu_1(1-\nu_2)\inf\pi_2 - (1-\sigma_1+\sigma_1\tau)(1-\tau)(\pi_2-\nu_2\inf\pi_2) \\
&= (1-\sigma_1+\sigma_1\tau)(1-\tau)(\mathbb{E}\pi_2-\pi_2).
\end{aligned} \tag{38}
$$

Lastly, the curve of constant marginal value of external debt finance of Firm 1,

$$
\lambda_{\nu_1} = \overline{\lambda}_{\nu_1}(k_1^*(\nu_1,k_2^C,\nu_2(\nu_1,k_2^C),k_2(k_2^C,\nu_2)),a'(\nu_1,\nu_2(\nu_1,k_2^C),k_1^*,k_2(k_2^C,\nu_2))),
$$

is derived by solving for the following system of equations with taking $dk_1^*=dk_2=da'=0$

$$
\begin{aligned}
A_1^{-1}dk_1^*(\nu_1,k_2^C,\nu_2) &= -dk_2^C + \frac{1-\tau}{1+r}(1-\sigma_1)\left\{\inf\pi_1 + (1-\sigma_1)(1-\tau)(1-\nu_2)\inf\pi_2\right\}d\nu_1 \\
&\quad + \frac{(1-\tau)^2(1-\sigma_2)}{1+r}\inf\pi_2\left\{-\sigma_1-(1-\sigma_1)\nu_1\right\}d\nu_2 \\
A_2^{-1}dk_2(k_2^C,\nu_2) &= dk_2^C + \frac{1-\tau}{1+r}(1-\sigma_2)\inf\pi_2 d\nu_2 \\
da'(\nu_1,\nu_2) &= -(1-\sigma_1)(1-\tau)\left\{\inf\pi_1 + (1-\sigma_2)(1-\nu_2)(1-\tau)\inf\pi_2\right\}d\nu_1 \\
&\quad - (1-\sigma_2)(1-\tau)\inf\pi_2\left\{(1-\sigma_1+\sigma_1\tau)-(1-\sigma_1)(1-\tau)\nu_1\right\}d\nu_2
\end{aligned} \tag{39}
$$

such that

$$
\begin{aligned}
dk_2^C &= \frac{\frac{1-\tau}{1+r}(1-\sigma_1)\left\{\inf\pi_1 + (1-\sigma_2)(1-\nu_2)(1-\tau)\inf\pi_2\right\}}{\tau + (1-\sigma_1)(1-\nu_1)(1-\tau)}d\nu_1 \\
&= -\frac{1-\tau}{1+r}(1-\sigma_2)\inf\pi_2 d\nu_2.
\end{aligned} \tag{40}
$$

Note that $da'=0$ is redundant with $dk_1^*=dk_2=0$.